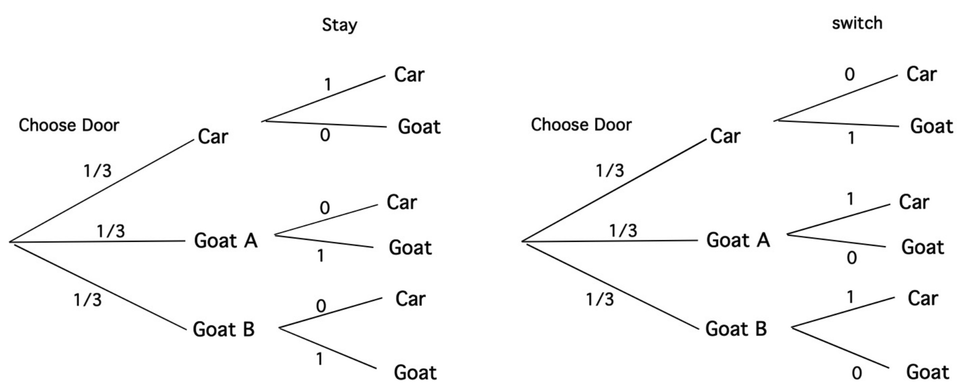
Two Person Games (Strictly Determined Games)

We have already considered how probability and expected value can be used as decision making tools for choosing a strategy. We include two examples below for review.

Example; The Monty Hall problem (Review: Using probability to decide on a strategy) This is a game where a player is asked to choose one of three doors. Behind one of the doors is a car or something worth winning (lets say a porsche) and behind the other two doors there is something unwanted or at least less desirable (perhaps a goat or plastic pink flamingoes for your lawn or a math book; lets go with the goats). The host knows exactly what is behind each door, the player does not. After the player chooses a door, the game show host opens one of the remaining doors, revealing a goat. The player is then allowed to switch doors or stick with their original choice. When the player has made their choice, the chosen door is opened and the player wins whatever is behind the door. In this game the player must choose a **strategy**; do they **switch** doors **or stay** with the original choice?

We can use probability as a decision making tool here and choose the strategy that maximizes our probability of success. This decision making tool makes most sense if a player is playing the game repeatedly or if they are part of a pool of players playing the game repeatedly. We see below that the probability of winning the car is 2/3 if the player switches and just 1/3 if the player stays with their original choice. Thus using probability as a decision making tool, we would choose to switch doors.
Below,we show the tree diagram labelled with probabilities associated to each strategy. The player's winnings are shown at the end of each path.



Example; Roulette (Review: Using expected value to decide on strategy) We saw earlier that in a game of American roulette where you bet \$1 on red, the probability distribution for your earnings (X) is given by:

X	**P(X)**
1	18/38
-1	20/38

The expected earnings for this game are $E(X) = -2/38$. Because the expected earnings are negative, we see that the player's earnings would average out as a loss in the long run and we would choose not to play this game repeatedly.

In general when **choosing a strategy**, it is better to choose a strategy which gives a higher probability of winning or a higher expected pay-off. However when choosing a strategy for game theory, our opponent is no longer chance, we have an opponent who

1. Wishes to maximize their own gain (which for a two person zero-sum game, equates to minimizing our gain),
2. Has full knowledge of the consequences of each strategy from the pay-off matrix, and assumes that we will play to our own advantage,
3. Can use statistics to notice patterns in our play if we are repeatedly playing the game and can anticipate the logical steps we might take to change our strategy and improve our pay-off.

The situation is thus much more dynamic and we need to take into account how our opponent is likely to play and how they will respond to any particular strategy we choose for repeated play. It will take us some time to describe how to find the best strategy for (some) two player zero-sum games.

Strictly Determined Games

We will start with a special type of game called a **strictly determined game** where the best strategy for both players will be to choose a single strategy/option and stick with it on every play (sometimes called a fixed or a pure strategy).

Example: From Last Lecture Roger and Colleen play a game. Each one has a coin. They will both show a side of their coin simultaneously. If both show heads, no money will be exchanged. If Roger shows heads and Colleen shows tails then Colleen will give Roger \$1. If Roger shows tails and Colleen shows heads, then Roger will pay Colleen \$1. If both show tails, then Colleen will give Roger \$2.

The pay-off matrix for Roger in this case is :

$$
\begin{array}{cc|cc}
 & & \multicolumn{2}{c}{\text{Colleen}} \\
R & & H & T \\ \hline
o & H & 0 & 1 \\
g. & T & -1 & 2
\end{array}
$$

When Roger is deciding on a strategy here, one might think that he should play tails since that offers the possibility of the maximum payoff, 2. However on second thoughts one realizes that if Colleen has full knowledge of the payoff matrix, she will never play tails because no mater what Roger plays Colleen always has a greater payoff by playing heads (Tails is called a **dominated strategy** for Colleen). So since Colleen is certain to play Heads, Roger should play Heads, (otherwise he will lose a dollar) and his payoff is zero. When both players play Heads, neither player can gain by changing their strategy. Thus the combination Rog: H and Coll: H is a point of equilibrium and we say that the matrix has a **saddle point** at Row 1, Col 1. The entry of the matrix at this point (= 0 = payoff for R for this combination of strategies) is called the **value of the game**.

In general when a player is contemplating what their best fixed strategy might be they keep three things in mind:

- They wish to maximize their payoff,
- Each player has full knowledge of the payoff matrix,
- Their opponent will play intelligently and wishes to maximize their own payoff.

Note that in a zero sum game The column player maximizes their payoff by minimizing the row players payoff.

Assume we have two players, R and C, where R has m strategies $r_1, r_2, \ldots, r_m$ and C has n strategies $c_1, c_2, \ldots, c_n$ where the payoff matrix is the $m \times n$ matrix shown below:

$$\begin{array}{cc|cccc} & \multicolumn{5}{c}{\mathbf{C}} \\ & & c_1 & c_2 & \ldots & c_n \\ \hline & r_1 & a_{11} & a_{12} & \ldots & a_{1n} \\ & r_2 & a_{21} & a_{22} & \ldots & a_{2n} \\ \mathbf{R} & \vdots & \vdots & \vdots & \vdots & \vdots \\ & r_m & a_{m1} & a_{m2} & \ldots & a_{mn} \end{array}$$

If R is going to play the same fixed strategy repeatedly, we can assume that C will minimize R's payoff by choosing the strategy (column) which corresponds to the minimum payoff (for R) in the row. Thus for each row of the payoff matrix, R can assume that their payoff will be the minimum entry in that row if they choose that strategy. To find the **Optimal fixed/pure strategy for R**:

(1) For each row of the pay-off matrix (R's Pay-off matrix), find the least element.

$$\begin{array}{cc|cccc|c} & \multicolumn{6}{c}{\mathbf{C}} \\ & & c_1 & c_2 & \ldots & c_n & \text{Min} \\ \hline & r_1 & a_{11} & a_{12} & \ldots & a_{1n} & \text{min of row 1} \\ & r_2 & a_{21} & a_{22} & \ldots & a_{2n} & \text{min of row 2} \\ \mathbf{R} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & r_m & a_{m1} & a_{m2} & \ldots & a_{mn} & \text{min of row m} \end{array}$$

(2) Choose the row for which this element is as large as possible (the row corresponding to the maximum of the numbers in the right hand column above).

Example Suppose Cat and Rat play a game where the pay-off matrix for Rat is given by:

	C_1	C_2	C_3
R_1	1	-1	0
R_2	6	-3	1
R_3	2	2	1

Find the optimal fixed strategy for R.

	C_1	C_2	C_3	*min*
R_1	1	-1	0	-1
R_2	6	-3	1	-3
R_3	2	2	1	1

Hence the optimal fixed strategy for R is R_3.

Similarly if C is searching for an optimal fixed strategy, they know that no matter which strategy (column) they settle on, the row player will choose the row which gives the maximum payoff for R. Thus to find the **Optimal fixed/pure strategy for C**:

(1) For each column of the pay-off matrix (R's Pay-off matrix), find the largest element.

C

		c_1	c_2	$\dots$	c_n
	r_1	a_{11}	a_{12}	$\dots$	a_{1n}
	r_2	a_{21}	a_{22}	$\dots$	a_{2n}
R	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	r_m	a_{m1}	a_{m2}	$\dots$	a_{mn}
	Max	max. of Col 1	max. of Col 2	$\dots$	max. of Col n

(2) Choose the column for which this element is as small as possible (the column corresponding to the minimum of the numbers in the bottom row above).

Example Find the optimal fixed strategy for Cat in the above game.

	C_1	C_2	C_3
R_1	1	-1	0
R_2	6	-3	1
R_3	2	2	1

	C_1	C_2	C_3
R_1	1	-1	0
R_2	6	-3	1
R_3	2	2	1
max	6	2	1

Hence the optimal fixed strategy for C is C_3.

If the largest of the row minima and the smallest of the column maxima occur at the same entry of the payoff matrix, then we say that the matrix has a **saddle point** at that location. In this case, we say the game is **strictly determined** and the value of the matrix entry at the saddle point is called **the value of the game**. In this case, the best strategy (the one which gives the maximum expected payoff if the game is played repeatedly and both players play intelligently) for both players is to adopt the fixed strategies corresponding to the saddle point and these two strategies are called a **solution to the game**.

Example

	C_1	C_2	C_3
R_1	1	-1	0
R_2	6	-3	1
R_3	2	2	1

(a) Is the game from the previous example between Cat and Rat a strictly determined game?

(b) If so what is the saddle point and the solution to the game?

(c) What is the value of the game?

(a) This game is strictly determined since the maximum row minima and the minimum column maximum both occur at position 3 3.

(b) Said another way the game has a saddle point which occurs at position 3 3.

(c) The value of the game is 1.

Example Catherine (C) and Rasputin (R) play a 2-player zero-sum game, where the payoff matrix for Rasputin is given by the following matrix:

	$C1$	$C2$	$C3$
$R1$	0	1	4
$R2$	5	-1	3
$R3$	3	2	5

(a) Is this a strictly determined game?

(b) If so where is the saddle point for the game? $(r, c) =$ ______.

What is the value of the game?

What is the solution to the game? Row ___ Column ___.

(c) Sometimes in reality players do not play optimally: If Rasputin always plays R1, which column should Catherine play in order to maximize her gain?

	$C1$	$C2$	$C3$	min
$R1$	0	1	4	0
$R2$	5	-1	3	-1
$R3$	3	2	5	2
max	5	2	5	

This game is strictly determined since the maximum row minima and the minimum column maximum both occur at position 3 2, or $(r, c) = (3, 2)$.

The value of the game is 2. The solution to the game is Row <u>3</u> Column <u>2</u>

If Rasputin can be counted on to play R1, Catherine should counter by playing C1, decreasing her payout from 2 to 0.

Note There may be more than one saddle point in a payoff matrix, in which case, there is more than one possible solution. However, in this case the value of the entry at each possible solution is the same and thus the value of the game is unique. The game is still considered to be strictly determined with a unique value, however the players may have more than one option for an optimal strategy.

Example Romeo (R) and Collette (C) play a zero-sum game for which the payoff matrix for Romeo is given by:

	$C1$	$C2$	$C3$	$C4$	$C5$
$R1$	1	5	9	1	4
$R2$	-3	-1	-3	-2	7
$R3$	-2	-3	1	-9	8
$R4$	1	2	2	1	14

(a) Find all saddle points for this matrix.

(b) What is the value of the game?

(c) What are the possible solutions to the game?

	$C1$	$C2$	$C3$	$C4$	$C5$	min
$R1$	1	5	9	1	4	1
$R2$	-3	-1	-3	-2	7	-3
$R3$	-2	-3	1	-9	8	-9
$R4$	1	2	2	1	14	1
max	1	5	9	1	14	

There are four saddle points: at $(1,1)$, $(1,4)$, $(4,1)$ and $(4,4)$. At each of these saddle points the payoff is 1 so the value of the game is 1.

Each player has 2 optimal strategies. Collette can play either C1 or C4. No matter which strategy Collette plays, Romeo can play either R1 or R4, so there are 4 solutions to the game $(1,1)$, $(1,4)$, $(4,1)$ and $(4,4)$.

Note Not every game is strictly determined, in which case a fixed strategy for both players is not optimal. We will discuss mixed strategies and optimal strategies for these games in the next two lectures. Meanwhile, here are some examples. **Example** Which of the games from the examples given in the previous lecture are strictly determined? If so, where is the saddle point and what is the value of the game?

		Colleen	
R		H	T
o	H	0	1
g.	T	-1	2

		C. attacks	
		B	F
R.	B	80%	100%
places bomb	F	90%	50%

		Charlie			
		$(1,2)$	$(1,3)$	$(2,3)$	$(2,4)$
R	$(1,2)$	0	2	-3	0
u	$(1,3)$	-2	0	0	3
t	$(2,3)$	3	0	0	-4
h	$(2,4)$	0	-3	4	0

		Colleen		
		H	T	*min*
Rog.	H	0	1	0
	T	-1	2	-1
max		0	2	

This game has a saddle point at $(1, 1)$. Hence it is strictly determined and has a value of 0.

		C. attacks		
		B	F	*min*
R. places bomb	B	80%	100%	80%
	F	90%	50%	50%
max		90%	100%	

The minimum of the maxima is 90 and the maxima of the minima is 80 so there is no saddle point.

		Charlie				
		$(1,2)$	$(1,3)$	$(2,3)$	$(2,4)$	*min*
R	$(1,2)$	0	2	-3	0	-3
u	$(1,3)$	-2	0	0	3	-2
t	$(2,3)$	3	0	0	-4	-4
h	$(2,4)$	0	-3	4	0	-3
max		3	2	4	3	

The minimum of the maxima is 2 and the maxima of the minima is 2 so there is no saddle point.

Example For the following payoff matrices, find the saddle points, values and solutions if they exist.

$$\begin{bmatrix} -1 & -3 & 3 \\ 5 & 0 & 2 \\ 6 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 4 & 3 \\ 5 & 2 & 2 \\ 6 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 5 & 2 \\ 6 & -2 \end{bmatrix} \qquad \begin{bmatrix} -1 & 1 \\ -5 & 2 \\ -1 & 2 \end{bmatrix}$$

-1	-3	3		-3
5	0	2		0
6	-1	1		-1
6	0	3		

Saddle point at $(2, 2)$ with game value 0.

-1	4	3		-1
5	2	2		2
6	-1	1		-1
6	4	3		

The minimum of the maxima is 3 and the maxima of the minima is 2 so there is no saddle point.

-1	1		-1
5	2		2
6	-2		-2
6	2		

Saddle point at $(2, 2)$ with game value 2.

-1	1		-1
-5	2		-5
-1	2		-1
-1	2		

The minimum of the maxima is -1 and the maxima of the minima is -1 so there is a saddle point(s). Saddle points at $(1, 1)$ and $(3, 1)$ with game value -1.